A quantity will now be determined which lies between I_n and S_n and is independent of n. It is clear that $x_{i+1}^m, x_{i+1}^{m-1}x_i, x_{i+1}^{m-2}x_i^2, \cdots, x_i^m$ form a decreasing sequence of $m+1$ positive terms and hence their arithmetic mean is greater than the smallest term x_i^m. Using this inequality it follows that

$$I_n < \Sigma(x_{i+1} - x_i)\frac{x_{i+1}^m + x_{i+1}^{m-1}x_i + x_{i+1}^{m-2}x_i^2 + \cdots + x_i^m}{m+1},$$

$$< \frac{1}{m+1}\Sigma(x_{i+1}^{m+1} - x_i^{m+1}) = \frac{b^{m+1}}{m+1} - \frac{a^{m+1}}{m+1}.$$

In a similar manner an inequality is found for S_n, and hence

$$0 < I_n < \frac{b^{m+1}}{m+1} - \frac{a^{m+1}}{m+1} < S_n. \tag{3}$$

By representing the three quantities $I_n, b^{m+1}/(m+1) - a^{m+1}/(m+1), S_n$ by points on a straight line and by considering the meaning of (2) and (3) as applied to these points, it will be obvious that the common limit of I_n and S_n is $b^{m+1}/(m+1) - a^{m+1}/(m+1)$. This then is the expression for the desired area. If the equation of the curve is $y = px^m$, it will be readily seen that the above result must be multiplied by p.

The same method, with a slight amount of extra manipulation, may be used for negative values of m and also for fractional values, excepting, however, the special case $m = -1$.[1]

An elementary evaluation of the area of any segment of an ordinary parabola by means of special properties of the curve is given in the *Traité de Géométrie* by Rouché et Comberousse, 2d vol., p. 348 (8th ed., 1912). The properties here used are such as might be given in the ordinary text on analytics. A somewhat similar treatment occurs in the first volume of Goursat-Hedrick's *Mathematical Analysis*, p. 134, and is referred to as one of the processes used by Archimedes. Here the summation of a geometric series is employed, but this may be avoided and the proof simplified by comparing the areas of the interior triangles with certain corresponding exterior triangles. These two proofs are somewhat similar to the one employed by Professor Moritz.

II. Inverse Trigonometric Functions.

By W. V. Lovitt, Colorado College.

As I look over the available text-books on trigonometry the feeling grows upon me that they are hastily written and some topics inadequately treated. It is certain that many errors are present.

I have examined twenty-four different texts with special reference to their treatment of the inverse trigonometric functions. In five the treatment was so

[1] An elementary treatment of this case was given by the writer under the title, A Geometric Treatment of the Exponential Function, in *Washington University Studies*, scientific series, vol. 6, no. 2, p. 33.

short (less than a page) that freedom from a positive error might be attributed to shortness. There are three which, although their treatment is also short (about two pages) are entirely correct and give positive warning against some of the errors which some of the others commit. They do not have a single example or exercise which could be criticized. Six more are to be commended in that what they do say in way of exposition is correct. They neglect however to point out a single pitfall and some of their exercises mislead the student. For instance, prove that

$$\arccos (1 - 2m^2) \equiv 2 \arcsin m.$$

They fail to point out that the expressions on opposite sides of the identity symbol do not represent the same sequence of angles; or that if we confine ourselves to the principal values of the angles the identity does not hold for all values of m, e.g. $m = -\frac{1}{2}$. It must be understood that this equation is true only for particular choices of the various possible values of the functions.

The remaining ten have positive errors. To quote from one author without naming him "the principal value is the one to be used in the following examples." Among the examples I find: Prove

$$\arccos (-4/7) = \arctan (-\sqrt{33}/4).$$

Nine authors make this same direct statement that the principal value only is to be used in a given list of problems. From their lists of problems I select five more, no two from the same author

$$2 \arctan 2 = \arcsin 4/5,$$

$$\arcsin 1/2 + \arccos \sqrt{3}/2 + \arctan \sqrt{3} = \arcsin \sqrt{3}/2,$$

$$2 \arctan x = \arccos (1 - x^2)/(1 + x^2) \quad [\text{try } x = 2]^1,$$

$$2 \arccos x = \arcsin (2x \sqrt{1 - x^2}) \quad [\text{try } x = 1/2],$$

$$\arctan m/n - \arctan (m - n)/(m + n) = \pi/4 \quad [\text{try } m = 2, n = -1].$$

Six of the nine ask the student to prove that

$$(1) \qquad \arctan m + \arctan n = \arctan (m + n)/(1 - mn).$$

But (1) is not true for principal values if $\arctan m + \arctan n$ is less than $-\pi/2$ or greater than $+\pi/2$. It is not true for instance when $m = n = +\sqrt{3}$.

The tenth man, who is cautious, says that "it is customary to limit the values of an inverse circular function to the smallest value." He asks us about a half page later to prove (1) and also to prove

$$(2) \qquad \arcsin 3/5 + \arcsin 8/17 = \arcsin 77/85.$$

If the principal values only are to be used then (1) is not always true as has already been pointed out. If other values are allowed (1) is true only for particular choices of the values of the functions involved. The latter remark applies equally

[1] Suggestions in brackets are my own.

well to equation (2). For example, (2) is not true if arcsin 3/5 and arcsin 8/17 are angles whose terminal lines lie in the second quadrant.

Direction is given in some instances to prove the statement (1) by taking the tangent of both sides. Without some restrictions on the angles involved this does not prove the equality; else

$$60^\circ = 240^\circ$$

because it happens that

$$\tan 60^\circ = \tan 240^\circ.$$

Is it any longer a mystery why the student does not obtain a better grasp of his mathematics?

III. The Path of a Projectile when the Resistance Varies as the Velocity.

By E. L. Rees, University of Kentucky.

The very simple problem of finding the path (and its hodograph) of a projectile in a vacuum is treated in most elementary treatises on vector analysis, but the problem of finding the trajectory when the body moves in a resisting medium is usually not considered as the general problem does not lend itself readily to vector treatment. However, there is a special case of some interest which presents no difficulties. If we assume that the resistance varies as the velocity (which under certain conditions is approximately true for low velocities in the air) the vector differential equation of motion is of a very simple type and its solution differs in no essential way from that of a scalar differential equation of the same type. The equation of such a motion is

$$\frac{d^2\boldsymbol{r}}{dt^2} = \boldsymbol{g} - k\frac{d\boldsymbol{r}}{dt},$$

where $\boldsymbol{g}$ is the vector acceleration due to gravity, and k is a positive scalar const. This is a linear vector differential equation which may be integrated at once by applying the integrating factor e^{kt}.

Integrating we get

$$\frac{d\boldsymbol{r}}{dt} = \frac{\boldsymbol{g}}{k} + e^{-kt}\boldsymbol{C}_1$$

and

$$\boldsymbol{r} = \frac{\boldsymbol{g}t}{k} - \frac{e^{-kt}}{k}\boldsymbol{C}_1 + \boldsymbol{C}_2,$$

where $\boldsymbol{C}_1$ and $\boldsymbol{C}_2$ are the vector constants of integration.

Assuming $\boldsymbol{r} = 0$, $\boldsymbol{v} = \boldsymbol{v}_0$ when $t = 0$ we have

$$\frac{d\boldsymbol{r}}{dt} = \frac{\boldsymbol{g}}{k} + e^{-kt}(\boldsymbol{v}_0 - \boldsymbol{g}/k) \tag{1}$$

and

$$\boldsymbol{r} = \frac{\boldsymbol{g}t}{k} + \frac{1}{k}(1 - e^{-kt})(\boldsymbol{v}_0 - \boldsymbol{g}/k). \tag{2}$$